VARIATION OF PLASMA RESISTIVITY DURING TOKAMAK START-UP

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The variation of plasma resistivity, due to collisions, during TOKAMAK start-up is reviewed. For the breakdown phase, Townsend Avalanche model is used and for the Coulomb phase, Spitzer calculations are taken into account. Anisotropic nature of the plasma is discussed through the resistivity tensor. Finally, by modeling the TOKAMAK start-up as a low-pressure inductively coupled discharge, the variation in plasma resistivity is derived. It is found that the plasma resistivity varies directly with the square of the reflected impedance and inversely with the square of the mutual inductance and oscillating frequency of the primary circuit.

I. INTRODUCTION

On the basis of resistivity, plasma has two types: collisional and collisionless. In the latter, collisions are so infrequent compared with any relevant variation in the fields or particle dynamics that they can safely be neglected. Most space plasmas belong to this type. In the former collisions are sufficiently frequent to influence or even dominate the behavior of the plasma.

Collisional plasma can again be divided into two classes: partially ionized plasma and fully ionized plasma. Partially ionized plasma contains a large amount of residual neutral atoms or molecules; while fully ionized plasma mainly consists of electrons and many types of ions, depending on the atom. It is clear that the types of collisions in both cases must be very different because neutrals do not respond to Coulomb fields. Taking into account the TOKAMAK start-up, the “Breakdown Phase” can be considered as the partially ionized plasma in which the direct collisions between the charge carriers and neutrals dominate whereas the “Coulomb Phase” can be considered as the fully ionized plasma in which the direct collisions are replaced by Coulomb collisions [1].

The outline of the article is as follows: A simple relation for the classical plasma resistivity is derived in Sec. II. Sec. III and IV are about the resistivity of partially ionized and that of fully ionized plasmas respectively. The anisotropic nature of the plasma resistivity is described in Sec. V. Sec. VI deals with a transformer model of an inductively-coupled discharge for finding out the reflected impedance that explains the plasma resistivity variation in TOKAMAK. Finally, the future prospects are presented in Sec. VII.

II. CLASSICAL PLASMA RESISTIVITY

In this section, an expression for classical plasma resistivity is derived by discussing the motion of plasma particles. In the presence of collisions, for an electron moving with velocity \( \mathbf{v}_e \) under the action of the Coulomb and Lorentz forces, the equation of motion is given by;

\[
m \frac{d\mathbf{v}_e}{dt} = -e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - m\mathbf{v}_e (\mathbf{v}_e - \mathbf{u})
\]

(1)

Where, \( \nu_c \) is the collision frequency.

Considering all the collision partners (ions in the case of a fully ionized or neutrals in a partially ionized plasma) at rest i.e. \( \mathbf{u} = 0 \), avoiding the anisotropic nature of the magnetized plasma by taking the electron velocity \( \mathbf{v}_e \) only in the direction of magnetic field \( \mathbf{B} \) i.e. \( \mathbf{v}_e \parallel \mathbf{B} \) and finally taking the steady state by assuming that the Coulomb force and the frictional drag balance each other, we get;

\[
\mathbf{E} = -\left( m \nu_c / e \right) \mathbf{v}_e
\]

(2)
Also, by the definition of current density;

\[ J = -en_v \]  \hspace{1cm} (3)

Using Eq. (3) in Eq. (2), we get;

\[ E = \left( m_v / n_e e^2 \right) J \]  \hspace{1cm} (4)

This is the familiar Ohm's law and defines the plasma resistivity '\( \eta || \) ' as;

\[ \eta || = \frac{m_e v}{n_e e^2} \]  \hspace{1cm} (5)

III. RESISTIVITY OF PARTIALLY IONIZED PLASMA

In partially ionized plasma most collisions occur between charged and neutral particles. Neutral particles affect the motion of charged particles by their mere presence as heavy compact obstacles. Hence, collisions between a charge and a neutral can be treated as head-on. Such a collision occurs only when a charged particle directly hits a neutral atom or molecule along its orbit. The neutral collision frequency is therefore given by;

\[ v_{ce} = n_v \sigma_n \langle v_e \rangle \]  \hspace{1cm} (6)

Where, \( n_n \) is the neutral particle density and \( \sigma_n \), the neutral collisional cross-section is given by;

\[ \sigma_n = \pi d_0^2 \]  \hspace{1cm} (7)

Typically, corresponding to the effective radius '\( d_0 \)' of the neutral particle, \( \sigma_n \approx 10^{-20} \text{m}^2 - 10^{-19} \text{m}^2 \).

Using Eqs. (6) and (7) in Eq. (5), we get;

\[ \eta || = \left( \frac{m_v \sigma_n}{e} \right) \left( \frac{n_n}{n_e} \right) \langle v_e \rangle \]  \hspace{1cm} (8)

Where, the 1st term in bracket is a constant, the 2nd term is the inverse degree of ionization and the last term is the average electron velocity that can be estimated by using Townsend Avalanche model.

The degree of ionization till the end of "Breakdown Phase" and before the start of "Coulomb Phase" can be approximated from Saha’s equation by the relation [2];

\[ \gamma \equiv \left( \frac{n_e}{n_n} \right) = \left[ 1 + \left( 5 \times 10^{-3} T_e^{3/2} \right)^{-1} \right]^{-1} \]  \hspace{1cm} (9)

Now using Townsend Avalanche model in which the average free electron is considered to acquire, after a few collisions, a constant drift speed \( \langle v_e \rangle (\text{ms}^{-1}) \) given by [4];

\[ \langle v_e \rangle = k \frac{E}{P} \]  \hspace{1cm} (10)

Where, \( E (\text{Vm}^{-1}) \) is the applied electric field, \( P (\text{torr}) \) is the gas pressure, \( k \) is the proportionality constant that depends on the gas used (for hydrogen, \( k \approx 43 \)) and the above relation is applicable for \( \frac{E}{P} \leq 2 \times 10^4 \text{ V.m}^{-1}.\text{torr}^{-1} \) because above this value runaway electrons may be produced [5].

IV. RESISTIVITY OF FULLY IONIZED PLASMA

In fully ionized plasmas the charged particles interact via their electric Coulomb fields. The existence of these fields implies that the particles are deflected at interparticle distances much larger than the atomic radius. The Coulomb potential therefore enhances the cross-section of the colliding particles, but
also leads to a preference for small angle deflections. Both these facts considerably complicate the calculation of a collision frequency in fully ionized plasma.

A further complication arises from the fact that in plasma with many particles in a Debye sphere, the Coulomb potential is screened and the electric field is approximately confined to the Debye sphere. One could think that the effective radius of the cross-section would become equal to the Debye radius, but this is not the case because the Coulomb deflections become increasingly smaller as the energy of the incident particle is increased and the Debye sphere is transparent for particles of sufficiently high energy. Since the potential increases steeply when approaching the center of the sphere, deflections will occur predominantly inside the Debye radius, but large angle deflections will still be rare. Formally, the Coulomb collision frequency in fully ionized plasma has the same functional dependence as Eq. (6). The problem lies in determining the Coulomb collisional cross-section, \( \sigma_c \).

\[
\nu_{el} = n_e \sigma_c \langle v_c \rangle \tag{11}
\]

In the following a simplified derivation of the Coulomb collision frequency between electrons and ions in fully ionized plasma is presented and later the modification introduced by the predominance of small angle deflections is included.

Consider the collision between a single heavy ion and an electron. Because of the much larger mass, the ion can be considered at rest. When the electron approaches the ion it will be deflected in the Coulomb field of the ion as shown in FIG. 1, due to its attraction toward the ion. In fully ionized plasma the temperature and consequently the energy of the electron is so high that the ion cannot trap the electron. The electron will turn around the ion and escape. Its orbit is a hyperbola which at large distances from the ion can be approximated by straight lines and close to the ion by a section of a circle of radius \( d_c \).

![FIG. 1. Electron orbit during a Coulomb collision with an ion.]

The distance \( d_c \) is called collision parameter or impact parameter. The simplest method to determine this quantity is to consider the Coulomb force an ion is exerting on an electron of mass \( m_e \), charge \( q = -e \), and velocity \( v_e \):

\[
F_c = -\frac{e^2}{4\pi\varepsilon_0 d_c^2} \tag{12}
\]

This force is felt by the electron only during an approximate average time \( \tau = d_c / v_e \), when it passes the ion. The change in momentum \( \Delta \left( m_e v_e \right) \) it experiences during this time is approximately given by:

\[
\left| \Delta \left( m_e v_e \right) \right| \approx \frac{d_c}{v_e} \frac{e^2}{4\pi\varepsilon_0 d_c^2} = \frac{e^2}{4\pi\varepsilon_0 v_e d_c} \tag{13}
\]
For large deflection angles $\gamma_e \approx 90^0$, the change in the particle momentum is of the same order as the momentum itself, i.e., $\Delta (m_e v_e) \approx m_e v_e$. Inserting this crude approximation in Eq. (13) enables us to determine $\sigma_c$ for a given velocity as:

$$\sigma_c = \pi d_c^2 \approx e^4 / 16\pi e_0^2 m_e^2 \langle v_e \rangle^4$$  \hspace{1cm} (14)

Where, $v_e$ is replaced by the average electron velocity $\langle v_e \rangle$ since the bulk of the electrons move at the average velocity. Using Eq. (14) in Eq. (11), we get:

$$v_{ei} = n_e \sigma_c \langle v_e \rangle \approx n_e e^4 / 16\pi e_0^2 m_e^2 \langle v_e \rangle^3$$  \hspace{1cm} (15)

Using the average thermal energy given by $KT_e = m_e \langle v_e \rangle^2$ for one dimension and applying the plasma frequency formula given by $\omega_{pe} = \sqrt{n_e e^2 / e_0 m_e}$, we get;

$$v_{ei} \approx \frac{1}{16\pi} \frac{\omega_{pe}}{n_e} \left( \frac{KT_e}{m_e} \right)^{3/2}$$  \hspace{1cm} (16)

This formula is not exact because there is no correction for the predominance of weak deflection angles as well as for the different velocities electrons assume in thermal equilibrium in the plasma. These corrections were made by Spitzer and Harm [7] who found that;

$$v_e \approx \frac{\omega_{pe}}{32\pi} \frac{\ln \Lambda}{\Lambda}$$  \hspace{1cm} (17)

Where, $\Lambda = \frac{b_{max}}{b_{min}} \approx \frac{\lambda_D}{b_{2/2}} \approx 4\pi n_e \lambda_D^3$ and $\ln \Lambda = \begin{cases} 16.34 + 1.5 \ln T_e - 0.5 \ln n_e & ; (T_e < 1.16 \times 10^5 \text{K}) \\ 22.81 + \ln T_e - 0.5 \ln n_e & ; (T_e > 1.16 \times 10^5 \text{K}) \end{cases}$

is known as the Coulomb Logarithm. Using Eq. (17) in Eq. (5) for $v_{ei}$, we get the Spitzer resistivity formula;

$$\eta_\parallel = \eta_\perp = \frac{m_e \omega_{pe} \ln \Lambda}{n_e e^2 32\pi \Lambda} \approx 2.8 \times 10^{-8} T_e^{-3/2}$$  \hspace{1cm} (18)

Where, $T_e$ is in keV, $\eta_\parallel$ is in $\Omega m$ and for TOKAMAKS, $\ln \Lambda = 17$. Eq. (18) shows that the Spitzer resistivity is independent from the plasma density, since $\omega_{pe}$ is directly and $\Lambda$ inversely proportional to the square root of the electron density. This can be verified from the fact that if one tries to increase the current by adding more charge carriers one also increases the collision frequency and the frictional drag and, by this, decreases the velocity of the charge carriers and, hence, the current.

V. ANISOTROPIC RESISTIVITY TENSOR

In this section, the anisotropic nature of the magnetized plasma is taken into account. The plasma resistivity $\eta$ is now a tensor of the form;

$$\eta = \begin{pmatrix} \sigma_p & \sigma_H & 0 \\ \sigma_H & -\sigma_p & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}^{-1}$$  \hspace{1cm} (19)
Where, the tensor elements are given by:

\[
\begin{align*}
\sigma_\parallel &= \sigma_0 = \eta_\parallel^{-1} \\
\sigma_P &= \frac{V_e^2}{V_e^2 + \omega_g^2} \eta_\parallel^{-1} \\
\sigma_H &= \frac{\omega_g V_e}{V_e^2 + \omega_g^2} \eta_\parallel^{-1}
\end{align*}
\]

(20)

The tensor element \(\sigma_\parallel\) is called parallel conductivity since it governs the magnetic field-aligned current driven by the parallel electric field component \(E_\parallel\). The parallel conductivity is equal to the plasma conductivity in the unmagnetized case. The element \(\sigma_P\) is called Pedersen conductivity and governs the Pedersen current in the direction of that part of the electric field \(E_\perp\) which is transverse to the magnetic field \(B\). The Hall conductivity \(\sigma_H\) determines the Hall current in the direction perpendicular to both the electric and magnetic field, in the \(-E \times B\) direction (note that \(\omega_g\) is a negative number). The dependence of the conductivity tensor elements on the ratio of the cyclotron frequency to the collision frequency is shown in FIG. 2.

![FIG. 2. Dependence of the Conductivities on the Gyro-to-Collision Frequency Ratio](image)

In a highly Collisional plasma containing a weak magnetic field, \(\omega_g \ll V_e\). The set of Eqs. (20) then shows that \(\sigma_\parallel = \sigma_0, \sigma_P \approx \sigma_0 \text{ and } \sigma_H = 0\), and the conductivity tensor becomes isotropic and reduces to a scalar but the Pedersen conductivity \(\sigma_P\) dominates, since in such a domain the electrons are scattered in the direction of the electric field before they can start to gyrate about the magnetic field. The conductivity is most anisotropic for plasmas with \(\omega_g \approx V_e\), since the electrons are scattered about once per gyration. Hence, both \(\mathbf{E} \times \mathbf{B}\) drift and motion along the transverse electric field are equally important and the Pedersen and Hall conductivities are of the same order. In this latter case, the electrons will, on average, move at an angle of 45° with both the direction of the transverse electric field and the \(\mathbf{E} \times \mathbf{B}\) direction. For a dilute, nearly Collisionless plasma with a strong magnetic field, \(\omega_g \gg V_e\) and the set of Eqs. (20) shows that \(\sigma_\parallel = \sigma_0, \sigma_P \approx 0\) and \(\sigma_H \neq 0\). Hence, in such plasma the current flows essentially along the field lines. The electrons experience the \(\mathbf{E} \times \mathbf{B}\) drift for many gyrocycles, before a collision occurs, and the Hall conductivity \(\sigma_H\) dominates.
VI. TRANSFORMER MODEL

In this section, TOKAMAK start-up is modeled as a low-pressure inductively coupled discharge in which the plasma is regarded as one turn secondary coil of an air-core transformer [8] as shown in FIG. 3.

![Electrical Circuit Representation of TOKAMAK Start-up as an Inductively-Coupled Discharge](image)

FIG. 3. Electrical Circuit Representation of TOKAMAK Start-up as an Inductively-Coupled Discharge

In this representation, the primary coil is modeled as central solenoid of the TOKAMAK with;

- \( V_1 \) = Charging Voltage of the Capacitor
- \( \omega \) = Oscillating Frequency of the Primary Circuit
- \( I_1 \) = Primary Coil Current
- \( L_1 \) = Primary Coil Inductance
- \( R_1 \) = Primary Coil Resistance

\[ R_2 = \text{Plasma Resistance} = \eta \frac{L}{A} = \eta \frac{2\pi R}{\pi a^2}; \quad \left\{ \begin{array}{l} R = \text{Major Radius} \\ a = \text{Minor Radius} \end{array} \right. \]

- \( I_2 \) = Secondary Coil Current
- \( \nu_c \) = Effective Electron Collision Frequency
- \( L_2 \) = Plasma Self-Inductance (the Geometric) due to the Discharge Current Path approximated by that of a Loop Conductor
- \( L_e \) = Electron Inertia Inductance = \( \frac{R_2}{\nu_c} \)
- \( L'_2 \) = Discharge Inductance = \( L_2 + L_e \)
- \( M \) = Mutual Inductance = \( k\sqrt{L_1 L_2} \)

- \( k \) = Coupling Coefficient
- \( n \) = Turn Ratio

So, applying KVL to the two meshes of the inductive discharge, we get;
Expressing $I_2$ in terms of $I_1$ from Eq. (22) and substituting it into Eq. (21), the input impedance $Z_m$ is given by;

$$Z_m = \frac{V_1}{I_1} = (R + j\omega L_1) + \left( \frac{\omega^2 M^2}{(R_2 - j\omega L_2)} \right)$$

(23)

Which is a combination of the primary impedance (1st term) and the reflected impedance (2nd term) due to the coupling between the primary and secondary circuits. From the point of view of the primary circuit, the effect of the coupled secondary circuit is to add this reflected impedance $Z_R$ to the primary circuit.

So, $Z_R = \frac{\omega^2 M^2}{(R_2 - j\omega L_2)} = \left( \frac{\omega^2 M^2}{(R_2 - j\omega (L_2 + L_c))} \right) = \left( \frac{\omega^2 M^2}{(R_2 - j\omega (L_2 + R_2/c))} \right)$

(24)

and

$$|Z_R| = Z_R = \omega^2 M^2 / \sqrt{R_2^2 + \omega^2 (L_2 + R_2/c)^2}$$

(25)

Since for an ideal transformer, the coupling is perfect, we may take $k = 1$ and $n = Z_R / \omega M$. So, seeing in the primary circuit, the variation in plasma resistivity $\eta_2$ is given by;

$$\eta_2 = \frac{Z_R}{\omega M} = \left( \frac{\omega^2 M^2}{(L_2 + R_2/c)^2 + R_2^2} \right) \eta_2$$

(26)

The above relation shows that the plasma resistivity $\eta_2$ changes with $\omega, M, L_2, R_2$ and $c$.

In the analysis presented here it is assumed that the discharge is in a purely inductive mode, implying that the capacitive mode of operation, which appears to dominate upon discharge initiation and at relatively low power, can be ignored and the power transfer to the plasma electrons can be represented by the plasma conductivity formula [8].

**VII. FUTURE WORK**

A future resistivity study with the inclusion of runaway electrons, neoclassical and anomalous effects would give a more complete picture of the various regimes of resistivity in the TOKAMAK. The runaway electrons have velocities up to the relativistic range, so result in low plasma resistivity which makes the Ohmic heating non-effective. The study of neoclassical resistivity due to the enhanced diffusion of the trapped particles in the magnetic islands (due to Banana orbits) and that of anomalous resistivity due to the plasma perturbations (not due to the collisions) are also aimed to include in the future work.